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Equation Guide

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Society of Actuaries
Exam MLC: Actuarial Models and Life Contingencies

Introduction:

This compilation of equations is designed to aid the student in preparing for Exam MLC (Actuarial Models or Mathematics) offered through the Society of Actuaries starting in the spring of 2007. The comprehensive list contains equations for most of the insurance- and annuity-related subjects that the exam covers. This formula sheet includes topics from life tables to reserves to counting distributions. The list is not guaranteed to cover all the material on the exam. Many nominal equations that are unlikely to show up on an exam are not included. The 650+ formulas are neatly organized into several categories. Refer to the table of contents for guidance.

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Section 2: Survival Equations

X = age at death = future lifetime for a person age 0
(x) = a person whose age is x years
ω = ultimate age = limiting age = age beyond which survival is impossible

$F_X(x) = \Pr(X \leq x)$
= probability that death will occur prior to or at age x

$f_X(t) =$ probability (x) dies exactly at age x+t = ${}_t p_x * \mu_x(t)$

$$F_X(x) = \int_0^x f_X(t) * dt$$

$$\int_0^\infty f_X(t)dt = 1 = F_X(\infty)$$

$s(x) =$ survival function
= 1 - $F_X(x)$
= probability death will not strike by age x
= probability of living up to at least age x

$$f_X(x) = -s'(x)$$

$$s(x) = \int_x^\infty f_X(t) * dt$$

$T(x)$ = future lifetime for a person age x

$$\Pr(x < X \leq z) = F_X(z) - F_X(x) = s(x) - s(z)$$

$$\Pr(x < X \leq z \mid X > x) = \frac{F_X(z) - F_X(x)}{1 - F_X(x)} = \frac{s(x) - s(z)}{s(x)}$$

${}_t q_x$ = probability (x) will die within t years
= probability (x) will die in the interval (x, x + t)
= $\Pr(T(x) \leq t)$

${}_t p_x$ = probability (x) will survive at least t years
= probability (x) will die in the interval (x + t, ω)
= $\Pr(T(x) > t)$
= $\dfrac{s(x+t)}{s(x)}$ = $p_x * p_{x+1} * p_{x+2} * \ldots * p_{x+t-1}$

$_0 p_x = 1$
$_x p_0 = s(x)$

$_{t|u} q_x$ = probability (x) survives at least t more years, but dies before reaching age (x + t + u)
$= \Pr(t < T(x) \le t + u)$
$= {}_{t+u} q_x - {}_t q_x = {}_t p_x - {}_{t+u} p_x = {}_t p_x * {}_u q_{x+t}$

$_t p_x = 1 - {}_t q_x$

$_t q_x = {}_1 q_x + {}_1 p_x * {}_1 q_{x+1} + {}_2 p_x * {}_1 q_{x+2} + {}_3 p_x * {}_1 q_{x+3} + ... + {}_{t-1} p_x * {}_1 q_{x+t-1}$
$= {}_{0|1} q_x + {}_{1|1} q_x + {}_{2|1} q_x + {}_{3|1} q_x + ... + {}_{t-1|1} q_x$

$_n E_x = v^n * {}_n p_x$

$K(x)$ = curtate future lifetime of (x) = greatest integer in T(x)
= number of whole years that (x) will live through in the future before dying
$\Pr(K(x) = k) = \Pr(k \le T(x) < k + 1) = {}_k p_x - {}_{k+1} p_x = {}_k p_x * q_{x+k} = {}_{k|} q_x$

$\mu(x)$ = force of mortality = $\dfrac{f_X(x)}{1 - F_X(x)} = \dfrac{-s'(x)}{s(x)} = \dfrac{f_X(x)}{s(x)} = \dfrac{-\frac{d}{dt}[{}_t p_x]}{{}_t p_x}$

If the force of mortality is multiplied by the constant c, then ${}_t p_x^{*} = ({}_t p_x)^c$
↑ Using $c * \mu_x(t)$ ↑ Using $\mu_x(t)$ original

<u>Constant Force of Mortality:</u>

$s(x) = e^{-\mu x}$
$_t p_x = e^{-\mu t}$
$\overset{o}{e}_x = \dfrac{1}{\mu}$
$Var(T(x)) = \dfrac{1}{\mu^2}$
$f_X(t) = \mu * e^{-\mu t}$
$_n E_x = e^{-(\mu + \delta) n}$

Function for Force of Mortality:

$$ {}_t p_x = e^{-\int_0^t \mu_x(r)\,dr} $$

Life Tables:

$l_0 = radix =$ number of people in the group at age 0

$l_x = l_0 * s(x) = l_0 * p_0 * p_1 * p_2 * .... * p_{x-1}$ = number of people in the group at age x

$l_{x+t} = l_x * {}_t p_x$

${}_n d_x =$ number of deaths between ages x and x+n = $l_x - l_{x+n} = l_x * {}_n q_x$

$q_x = \dfrac{l_x - l_{x+1}}{l_x} = \dfrac{d_x}{l_x}$ $\quad\quad$ ${}_n q_x = \dfrac{l_x - l_{x+n}}{l_x} = \dfrac{{}_n d_x}{l_x}$

$p_x = \dfrac{l_{x+1}}{l_x}$ $\quad\quad$ ${}_n p_x = \dfrac{l_{x+n}}{l_x}$

$[x] =$ a person selected at age x to have select status

$\overset{o}{e}_x =$ complete expectation of life = E(T(x)) = expected number of future years of life for (x)

= expected future lifetime of (x)

$$ = \int_0^\infty {}_t p_x \text{ dt} = \frac{T_x}{l_x} $$

$$ \text{Var(T(x))} = 2 * \int_0^\infty t * {}_t p_x \text{ dt} - \overset{o}{e}_x{}^2 $$

$m(x) =$ median future lifetime of (x)

$$ = \text{m such that } {}_m p_x = {}_m q_x = \frac{1}{2} $$

$e_x =$ curtate expectation of life = E(K(x))

$$ = \sum_{k=1}^\infty {}_k p_x $$

$$ \text{Var(K(x))} = \sum_{k=1}^\infty (2k-1) * {}_k p_x - e_x{}^2 $$

$L_x =$ expected total amount of years lived by members of the group between ages x and x+1

$$ = \int_0^1 l_{x+t} \text{ dt} $$

$${}_nL_x = \int_0^n l_{x+t}\ \text{dt}$$

$T_x =$ expected total amount of years lived by members of the group after age x

$$= \int_0^\infty l_{x+t}\ dt = {}_\infty L_x$$

$m_x =$ central death rate over the interval (x, x+1)

$$= \frac{l_x - l_{x+1}}{L_x} = \frac{d_x}{L_x}$$

$${}_nm_x = \frac{l_x - l_{x+n}}{{}_nL_x} = \frac{{}_nd_x}{{}_nL_x}$$

"death rate" = number of deaths per year of life

$\overset{o}{e}_{x:\overline{n}|}$ = n-year temporary complete life expectancy of (x)

$$= \frac{{}_nL_x}{l_x} = \int_0^n {}_tp_x\ \text{dt} = E(T \wedge n)$$

$e_{x:\overline{n}|}$ = n-year temporary curtate life expectancy of (x) = $\sum_{k=0}^{n} {}_kp_x = E(K \wedge n)$

Recursion: $\overset{o}{e}_x = \overset{o}{e}_{x:\overline{n}|} + {}_np_x * \overset{o}{e}_{x+n}$

$\overset{o}{e}_x = {}_1p_x + {}_1p_x * \overset{o}{e}_{x+1}$ and $e_x = {}_1p_x + {}_1p_x * e_{x+1}$

Linear Interpolation: $s(x+t) = (1-t)*s(x) + t*s(x+1)$, where $0 \le t \le 1$

Under a Uniform Distribution of Deaths (UDD):

$${}_tq_x = t *_1 q_x$$

$${}_yq_{x+t} = \frac{y*q_x}{1-t*q_x} = \frac{{}_yq_x}{{}_tp_x} = \Pr(T(x) < y \mid T(x) > t) \quad \text{(If } 0 < \text{t+y} < 1)$$

$$\mu_x(t) = \frac{q_x}{1-t*q_x} = \frac{q_x}{{}_tp_x}$$

$${}_tp_x * \mu(x+t) = q_x$$

$$\overset{o}{e}_x = e_x + \frac{1}{2} = E(T(x)) = E(K(x)) + \frac{1}{2}$$

$$Var(T(x)) = Var(K(x)) + \frac{1}{12}$$

Laws of Mortality Table:

	${}_t p_x$	$s(x)$	$\mu(x)$	$\mu_x(t)$	$\overset{o}{e}_x$	Var(T(x))	$f_X(x)$	$F_X(x)$	Other
DeMoivre	$\frac{\omega-x-t}{\omega-x}$	$\frac{\omega-x}{\omega}$	$\frac{1}{\omega-x}$	$\frac{1}{\omega-x-t}$	$\frac{\omega-x}{2}$	$\frac{(\omega-x)^2}{12}$	$\frac{1}{\omega}$	$\frac{x}{\omega}$	
Modified DeMoivre	$\left(\frac{\omega-x-t}{\omega-x}\right)^c$	$\left(\frac{\omega-x}{\omega}\right)^c$	$\frac{c}{\omega-x}$	$\frac{c}{\omega-x-t}$	$\frac{\omega-x}{c+1}$	$\frac{(\omega-x)^2 * c}{(c+1)^2 * (c+2)}$			
Gompertz		$e^{-m(c^x-1)}$	Bc^x						$m=\frac{B}{\ln(c)}$
Makeham		$e^{-Ax-m(c^x-1)}$	$A+Bc^x$						$m=\frac{B}{\ln(c)}$
Weibull		$e^{-ux^{n+1}}$	kx^n						$u=\frac{k}{n+1}$

Under DeMoivre's, $l_x=\omega-x$ $\quad e_x=\frac{\omega-x-1}{2}$ $\quad {}_t q_x=\frac{t}{\omega-x}$

<u>Section 3: APV of Life Insurance</u>

<u>Key Variables:</u>

b_t = benefit function

v^t = discount function

$z_t = b_t * v^t$ = present value of the benefit function

$Z = b_t * v^t$ = present value of benefits random variable

$E(Z)$ = APV of benefit

k = curtate value of t = complete years since issue of the policy

<u>Continuous Insurance:</u>

*Fully continuous insurance = Benefit is paid at the moment of death and the owner pays premiums continuously

*The bar over 'A' indicates that the benefit is paid at the moment of death

In general, for a life insurance spanning times a and b,

$$E(Z)=\int_a^b b_t * v^t * {}_t p_x * \mu_x(t)\ dt = \int_a^b b_t * v^t * f(t)\ dt$$

Whole Life Insurance:

$$b_t = 1$$
$$v_t = v^t$$
$$Z = v^T$$
$$E(Z) = \overline{A}_x = \int_0^\infty v^t * {}_t p_x \mu_x(t)\, dt = E(v^T)$$
$$Var(Z) = {}^2\overline{A}_x - \overline{A}_x^{\,2}$$

If μ and δ are constants, then $\overline{A}_x = \dfrac{\mu}{\mu+\delta}$ and ${}^2\overline{A}_x = \dfrac{\mu}{\mu+2\delta}$

Under DeMoivre's, $\overline{A}_x = \dfrac{1}{\omega - x} * \overline{a}_{\overline{\omega-x|}}$

n-year Term Life Insurance:

$$b_t = \begin{cases} 1, t \le n \\ 0, t > n \end{cases}$$
$$v_t = v^t$$
$$Z = \begin{cases} v^T, T \le n \\ 0, T > n \end{cases}$$
$$E(Z) = \overline{A}^{1}_{x:\overline{n|}} = \int_0^n v^t * {}_t p_x \mu_x(t)\, dt$$

If μ and δ are constants, then $\overline{A}^{1}_{x:\overline{n|}} = \dfrac{\mu}{\mu+\delta} * (1 - {}_nE_x)$

n-year Pure Endowment:

$$b_t = \begin{cases} 0, t < n \\ 1, t \ge n \end{cases}$$
$$v_t = v^n$$
$$Z = \begin{cases} 0, T < n \\ v^n, T \ge n \end{cases}$$
$$E(Z) = A_{x:\overline{n|}}^{\ \ \,1} = v^n * {}_n p_x = {}_nE_x$$

n-year Endowment:

$$b_t = 1$$
$$v_t = \begin{cases} v^t, t < n \\ v^n, t \ge n \end{cases}$$
$$Z = \begin{cases} v^T, T < n \\ v^n, T \ge n \end{cases}$$

$$E(Z) = \overline{A}_{x:\overline{n}|} = \overline{A}^{\,1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;1} = \int_0^n v^t *_t p_x \mu_x(t)\, dt + v^n *_n p_x$$

m-year Deferred Whole Life Insurance:

$$b_t = \begin{cases} 0, t < m \\ 1, t \geq m \end{cases}$$

$$v_t = v^t$$

$$Z = \begin{cases} 0, T < m \\ v^T, T \geq m \end{cases}$$

$$E(Z) = {}_{m|}\overline{A}_x = \int_m^\infty v^t *_t p_x \mu_x(t)\, dt = \overline{A}_x - \overline{A}^{\,1}_{x:\overline{m}|} = {}_m E_x * \overline{A}_{x+m}$$

$${}^2_{m|}\overline{A}_x = v^{2m} *_m p_x *{}^2\overline{A}_{x+m}$$

Whole Life Insurance with Benefit Increasing Annually:

$$b_t = \lfloor t+1 \rfloor$$

$$v_t = v^t$$

$$Z = \lfloor t+1 \rfloor * v^t$$

$$E(Z) = (I\overline{A})_x = \int_0^\infty \lfloor t+1 \rfloor v^t *_t p_x \mu_x(t)\, dt$$

Whole Life Insurance with Benefit Increasing mthly:

$$b_t = \left\lfloor \frac{t}{1/m} + 1 \right\rfloor$$

$$v_t = v^t$$

$$Z = \left\lfloor \frac{T}{1/m} + 1 \right\rfloor * v^T$$

$$E(Z) = (I^{(m)}\overline{A})_x = \int_0^\infty \left\lfloor \frac{t}{1/m} + 1 \right\rfloor v^t *_t p_x \mu_x(t)\, dt$$

If m approaches infinity,

$$b_t = t$$

$$v_t = v^t$$

$$Z = T * v^T$$

$$E(Z) = (\overline{IA})_x = \int_0^\infty t v^t *_t p_x \mu_x(t)\, dt$$

n-year Term Insurance with Benefit Decreasing by 1 Each Year:

$$b_t = \begin{cases} n - \lfloor t \rfloor, t \le n \\ 0, t > n \end{cases}$$

$$v_t = v^t$$

$$Z = \begin{cases} (n - \lfloor T \rfloor) v^T, T \le n \\ 0, T > n \end{cases}$$

$$E(Z) = (D\bar{A})^{1}_{x:\overline{n|}} = \int_0^n (n - \lfloor t \rfloor) v^t * {}_t p_x \mu_x(t)\, dt$$

For all types of insurance described above (excluding the increasing and decreasing types):

$Var(Z) = E(Z^2) - E(Z)^2$ = (2nd moment APV life insurance) – (1st moment APV life insurance)^2

$E(Z^j) = E(Z)$ with force of interest $\delta * j$

$E(Z^2) = E(Z)$ with all instances of the force of interest doubled; v^n becomes v^{2n}

Discrete Insurance:

*Fully discrete insurance = Benefit is paid at the end of the year of death and the owner pays a premium once per year

*The lack of a bar over 'A' indicates that benefits are paid at the end of the year of death

Whole Life Insurance:

$$b_{k+1} = 1$$

$$v_{k+1} = v^{k+1}$$

$$Z = v^{K+1}$$

$$E(Z) = A_x = \sum_{k=0}^{\infty} v^{k+1} * {}_k p_x * q_{x+k}$$

$$Var(Z) = {}^2A_x - A_x^{\ 2}$$

Recursion Equation:

$$A_x = vq_x + vp_x * A_{x+1} = vq_x + v^2 * p_x * q_{x+1} + v^2 *{}_2 p_x * A_{x+2}$$

$${}^2A_x = v^2 * q_x + v^2 * p_x *{}^2A_{x+1}$$

Under DeMoivre's, $A_x = \dfrac{1}{\omega - x} * a_{\overline{\omega - x|}}$

n-year Term Life Insurance:

$$b_{k+1} = \begin{cases} 1, 0 \le k \le n-1 \\ 0, k \ge n \end{cases}$$

$$v_{k+1} = v^{k+1}$$

$$Z = \begin{cases} v^{K+1}, 0 \le K \le n-1 \\ 0, K \ge n \end{cases}$$

$$E(Z) = A^{1}_{x:\overline{n|}} = \sum_{k=0}^{n-1} v^{k+1} * {}_k p_x * q_{x+k}$$

Recursion Equation:

$$A_{\overset{1}{x}:\overline{n}|} = vq_x + vp_x * A_{\overset{1}{x+1}:\overline{n-1}|}$$

n-year Pure Endowment:

$$A_{x:\overset{1}{\overline{n}|}} = v^n *_n p_x = {}_n E_x$$

n-year Endowment:

$$A_{x:\overline{n}|} = A_{\overset{1}{x}:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}} = \sum_{k=0}^{n-1} v^{k+1} *_k p_x * q_{x+k} + v^n {}_n p_x$$

m-year Deferred Whole Life Insurance:

$$_{m|}A_x = v^m *_m p_x * A_{x+m} = {}_m E_x * A_{x+m} = \sum_{k=m}^{\infty} v^{k+1} *_k p_x * q_{x+k}$$

<u>Other Life Insurance APV Equations:</u>

Generally, (APV at age x) < (APV at age x+n) for n>0

Generally, $\overline{A}_{**} > A_{**}$ for all types of life insurance

Under DeMoivre's, $_k p_x * q_{x+k} = \dfrac{1}{\omega - x}$

If UDD exists, $\overline{A}_x = \dfrac{i}{\delta} * A_x$, $A_x^{(m)} = \dfrac{i}{i^{(m)}} * A_x$, $\overline{A}_{\overset{1}{x}:\overline{n}|} = \dfrac{i}{\delta} * A_{\overset{1}{x}:\overline{n}|}$

$$A_x = A_{\overset{1}{x}:\overline{t}|} + {}_t E_x * A_{x+t}$$

$$^2\overline{A}_x = \int_0^{\infty} b_t^{\,2} * v^{2t} *_t p_x * \mu_x(t)\, dt$$

Under UDD, $^2\overline{A}_x = \dfrac{e^{2\delta} - 1}{2\delta} * {}^2A_x$